A LOWER BOUND ON THE BLOW UP RATE FOR THE DAVEY-STEWARTSON SYSTEM ON THE TORUS

NICOLAS GODET

ABSTRACT. We consider the hyperbolic-elliptic version of the Davey-Stewartson system with cubic nonlinearity posed on the two dimensional torus. A natural setting for studying blow up solutions for this equation takes place in $H^s, 1/2 < s < 1$. In this paper, we prove a lower bound on the blow up rate for these regularities.

1. Introduction

We consider the Davey-Stewartson system defined on the two dimensional torus $T^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$\left\{ \begin{array}{rcl} i\partial_t u - \partial_x^2 u + \partial_y^2 u &=& -|u|^2 u + 2u\partial_x \phi, \\ (\partial_x^2 + \partial_y^2)\phi &=& \partial_x |u|^2, \end{array} \right. \tag{1}$$

where $u : \mathbb{R} \times T^2 \to \mathbb{C}$ and $\phi : \mathbb{R} \times T^2 \to \mathbb{R}$ are the unknowns. Rearranging the second equation, we may see this system as a dispersive equation with an hyperbolic linear part and a nonlocal nonlinearity:

$$i\partial_t u + Pu = -|u|^2 u - E(|u|^2)u, \qquad (t, x) \in \mathbb{R} \times T^2, \tag{2}$$

where $P = -\partial_x^2 + \partial_y^2$ and E is the nonlocal operator such that:

$$\begin{array}{rcl} \widehat{E(f)}(m,n) &=& \dfrac{2m^2}{m^2+n^2}\widehat{f}(m,n), \qquad (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \\ \widehat{E(f)}(0,0) &=& 0. \end{array}$$

The Cauchy problem and the blow up theory for this equation have been studied essentially in the case where the system is posed in $\mathbb{R}^2$. In this case, the system is locally well posed in the Sobolev spaces L^2, H^1 (see [5]) and more easily for higher regularities $H^s, s > 1$. In [14], T. Ozawa proved that the equation posed on $\mathbb{R}^2$ enjoys a pseudo-conformal type symmetry and as for NLS, this allows to construct a blow-up solution by applying this transformation to an explicit stationnary (periodic in time for NLS) solution:

$$u(t, x, y) = \frac{1}{1 + x^2 + y^2}.$$

This function is then transformed into:

$$v(t,x,y) = \frac{1}{a+bt}\exp\left(\frac{ib}{4(a+bt)}(-x^2+y^2)\right)\frac{1}{1+\left(\frac{x}{a+bt}\right)^2+\left(\frac{y}{a+bt}\right)^2}, \tag{3}$$

with $(a,b)\in\mathbb{R}^2$. Note that $v(t)$ is in $H^s(\mathbb{R}^2)$ (see [14]) for every $s<1$ with $\|v(t)\|_{L^2}=\sqrt{\pi}$ but is not in $H^1(\mathbb{R}^2)$. The solution v blows up at time $T=-a/b$ in L^2 in the sense that the L^2 blow-up criteria is satisfied:

$$\|v\|_{L^4([0,t])L^4(\mathbb{R}^2)} \sim \frac{C}{(T-t)^{1/4}} \to \infty \text{ as } t \text{ goes to } T,$$

and accumulates all the mass in the origin:

$$|v(t)|^2 \to \pi\delta_{(0,0)} \qquad \text{as } t\to T \text{ in } \mathcal{D}'(\mathbb{R}^2).$$

The explosion also occurs in $H^s, s<1$ with the pseudo-conformal lower bound ([14]):

$$\|v(t)\|_{H^s} \sim \frac{C}{(T-t)^s}.$$

Note that in $\mathbb{R}^2$, we have a scaling symmetry; if u solves (2) then for all $\lambda>0, (t,x,y)\mapsto \lambda u(t,\lambda^2 t,\lambda x,\lambda y)$ also solves (2). It is a classical fact that this symmetry automatically implies a lower bound on the blow up rate: for all blow up solution u with maximal time $T(u)<\infty$, we have:

$$\|u(t)\|_{H^s} \geq \frac{C}{(T(u)-t)^{s/2}}. \tag{4}$$

By analogy with NLS in $\mathbb{R}^2$, we may ask the question of existence of ground states of the type $u(t,x,y)=\exp(i\omega t)Q(x,y)$ for (1) posed on $\mathbb{R}^2$ with an exponentially decaying profile Q but the hyperbolicity of the operator $-\partial_x^2+\partial_y^2$ forbids the existence of such solutions at least in the case where the nonlinearity is $-|u|^2u$ [6]. Moreover, numerics [11] seems to show that the L^2-norm of the solution u (or v) is the minimal mass for which we may have singularities in finite time. Thus, the function u plays the role of a ground state but is only polynomially decaying and this requires to work with low regularities H^s, $s<1$.

The aim of this paper is twofold: first give an H^s framework for studying blow up theory for (2) i.e. show the local well-posedness of (2) for initial data in $H^s(T^2)$ $s<1$ and secondly show that the lower bound on the blow up rate (4) still holds even if a scaling symmetry does not strictly make sense on the torus. The proof relies on local existence arguments on the dilated torus $\mathbb{R}^2/2\pi L\mathbb{Z}^2$, $L\to\infty$ and more precisely on bilinear Strichartz estimates. The classical method [2] giving well posedness from bilinear Strichartz estimates does not work in our setting because of the non local term; we will have to refine the bilinear approach with more general localizations. An interesting question, not solved here, is the localization of the solution v (3) i.e. construct from v a solution of (2). The non exponential decay of v is reflected in the estimate

$$\|v(t)\|_{H^s(\varepsilon\leq|(x,y)|\leq A)} \leq C(T-t)^{(1-s)},$$

which make perturbation arguments around v difficult to apply even on a compact domain. In particular, for $s>1$, blow up is not localized and this explains our choice to treat low regularities.

Remark 1. The system we study is called the hyperbolic-elliptic version of the generalized Davey-Stewartson system:

$$\left\{ \begin{array}{rcl} i\partial_t u + \varepsilon_1 \partial_x^2 u + \partial_y^2 u &=& -|u|^2 u + 2u\partial_x \phi, \\ (\varepsilon_2 \partial_x^2 + \partial_y^2)\phi &=& \partial_x |u|^2, \end{array} \right. \tag{5}$$

where $\varepsilon_i \in \{-1, +1\}$. Depending on the values of ε_i, the local well-posedness holds [5], [1], [13], [10], [4] but blow up theory is really well understood only in the elliptic-elliptic case $\varepsilon_1 = \varepsilon_2 = 1$ [12], [17] [15] where results are similar to those for NLS.

2. Statement and proof of the result

Let us now give our result.

Theorem 1. *Let $s > 1/2$.*

1) The equation (2) is locally well posed in the space $H^s(T^2)$ in the following sense: if $u_0 \in H^s(T^2)$ then there exist $b > 1/2$, a time $T > 0$ and a unique solution $u \in X_T^{s,b} \subset \mathcal{C}([0,T], H^s(T^2))$ satisfying (2). Here, $X_T^{s,b}$ denotes the Bourgain space associated to (2) and defined in (14).

2) If u_0 is an initial data such that its maximal time of existence $T(u_0)$ is finite then we have the lower bound: there exists $C(u_0) > 0$ such that for all $t \in [0, T(u_0))$,

$$\|u(t)\|_{H^s(T^2)} \geq \frac{C(u_0)}{(T(u_0) - t)^{s/2}}. \tag{6}$$

Strategy of the proof. The idea is to dilate the torus $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ by considering $T_L^2 = \mathbb{R}^2/2\pi L\mathbb{Z}^2$ where $L > 0$ will tend to infinity. In a first step, we perform a Banach fixed point argument in the dilated Bourgain space $X_{T,L}^{s,b}$ to obtain a bound on the blow up time of solutions of the equation in T_L^2:

$$T(u_0) \geq F(\|u_0\|_{H^s}), \tag{7}$$

for some function F independant of L. This relies on a uniform bilinear Strichartz estimate and a cutting of high frequencies to treat the nonlocal nonlinearity. In our analysis, it is of importance that dispersive estimates are local in space and time. In the step 2, we deduce the blow up lower bound from a scaling argument. The bound (6) which is the same as $\mathbb{R}^2$ is in accordance with the fact that when L goes to infinity T_L^2 looks like $\mathbb{R}^2$ formally. Note that machinery of Bourgain spaces is natural to study such questions but we do not exclude the possibility of working on other spaces by adapting harmonic analysis results to the case of T_L^2 to treat the operator E.

Notations. We denote by $e_{m,n}(x,y) = (2\pi)^{-1}\exp(imx + iny)$ the usual orthonormal basis of $L^2(T_1^2)$. When working on T_L^2, we will keep the same notation $e_{m,n}$ for the rescaled basis: $e_{m,n}(x,y) = (2\pi L)^{-1}\exp(i(m/L)x + i(n/L)y)$. For a function u defined on T_L^2, we note

$$\Delta_Q(u) = \sum_{(m,n)\in Q} c(m,n) e_{m,n},$$

where $c(m,n)$ are the Fourier coefficients of u:

$$c(m,n) = \frac{1}{2\pi L} \int_{T_L^2} u(x,y) e^{i(\frac{m}{L}x + \frac{n}{L}y)} dxdy.$$

If $Q \subset \mathbb{Z}^2$ and R is a dyadic number, we set for a function $u(t,x)$ defined on $\mathbb{R} \times T_L^2$:

$$\Delta_{Q,R} u = \sum_{(m,n)\in Q} \left(\int_{R \leq \langle \tau - \frac{m^2}{L^2} + \frac{n^2}{L^2} \rangle \leq 2R} \widehat{c_{n,m}}(\tau) e^{2i\pi t\tau} d\tau \right) e_{m,n},$$

where $c_{m,n}(t)$ are the Fourier coefficients of $u(t)$. When Q is the cube $Q = \{(m,n) \in \mathbb{Z}^2, N \leq \mathrm{Max}(|m/L|, |n/L|) \leq 2N\}$, we will note $\Delta_N = \Delta_Q$ and $\Delta_{N,R} = \Delta_{Q,R}$.

Proof. **Step 1.** We prove: for all $s > 1/2, L \geq 1$ and $u_0 \in H^s(T_L^2)$, there exists a solution u of

$$i\partial_t u + Pu = -|u|^2 u - E(|u|^2)u, \quad (x,y) \in T_L^2, \tag{8}$$

and $\alpha > 0$, $C > 0$ independant of L satisfying the lower bound on the blow up time:

$$T(u_0) \geq \frac{C}{\|u_0\|^{\alpha}_{H^s(T_L^2)}}. \tag{9}$$

The point here is that the lower bound depends only on the size of the initial data and not on L. On T_L^2, we denote (without changing notations) by P and E the natural extensions of the operators P and E defined above on T^2. Hence, symbols are respectively $(-m^2 + n^2)/L^2$ and $2m^2/(m^2+n^2)$.

High regularity. Before looking at low regularities and to convince the reader that 9) holds, let us focus on the easier case of more regular data i.e. $s \in \mathbb{N} \setminus \{0,1\}$. Let us prove (9) in this case. Let $L \geq 1$ and consider the equation (8) and its equivalent formulation

$$u(t) = e^{itP} u_0 + i \int_0^t e^{i(t-\tau)P} \left(|u(\tau)|^2 u(\tau) + E(|u(\tau)|^2) u(\tau) \right) d\tau. \tag{10}$$

Let $u_0 \in H^s(T_L^2)$. Taking the H^s-norm in (10) and using the triangle inequality, we get for a constant $C > 0$ independant of L:

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + C \int_0^T \left(\|u(\tau)^2 u(\tau)\|_{H^s} + \|E(|u(\tau)|^2) u(\tau)\|_{H^s} \right) d\tau. \tag{11}$$

Now we need a Sobolev type inequality with constants independant of the size of the torus.

Lemma 1. *There exists a constant $C > 0$ such that for all $L > 0$ and $v, w \in H^s(T_L)$,*

$$\|vw\|_{H^s(T_L^2)} \leq C \|u\|_{H^s(T_L^2)} \|v\|_{H^s(T_L^2)}.$$

Proof. We first prove that for all $s > 1$ and $v \in H^s(T_L)$, $\|v\|_{L^\infty} \leq C\|v\|_{H^s}$ for a constant $C > 0$ depending only on s. Indeed, expanding v in Fourier series, we first get

$$|v| \leq \frac{1}{2\pi L} \sum_{(m,n)\in\mathbb{Z}^2} |v_{m,n}|.$$

We make appear the H^s-norm of u and use Cauchy-Schwarz inequality to obtain:

$$\begin{aligned} \|v\|_{L^\infty} &\leq \frac{1}{2\pi L}\sum_{(m,n)\in\mathbb{Z}^2}|v_{m,n}|\left(1+\frac{m^2}{L^2}+\frac{n^2}{L^2}\right)^{s/2}\left(1+\frac{m^2}{L^2}+\frac{n^2}{L^2}\right)^{-s/2} \\ &\leq \frac{1}{2\pi L}\|v\|_{H^s}\left(\sum_{(m,n)\in\mathbb{Z}^2}\left(1+\frac{m^2}{L^2}+\frac{n^2}{L^2}\right)^{-s}\right)^{1/2}. \qquad (12) \end{aligned}$$

But we can easily compute the dependance in L of the last sum above by comparing with an integral as follow:

$$\begin{aligned} \sum_{(m,n)\in\mathbb{Z}^2}\left(1+\frac{m^2}{L^2}+\frac{n^2}{L^2}\right)^{-s} &\leq C\sum_{(m,n)\in\mathbb{Z}^2}\left(1+\frac{m^2}{L^2}\right)^{-s/2}\left(1+\frac{n^2}{L^2}\right)^{-s/2} \\ &\leq C\left(\int\frac{dx}{\left(1+\frac{x^2}{L^2}\right)^{s/2}}\right)^2 \\ &\leq CL^2. \end{aligned}$$

Thus, the dependance in L vanishes in (12) and we obtain the claim. Now we can prove the lemma by first writing the chain rule since $s\in\mathbb{N}$

$$\begin{aligned} \|(-\Delta)^{s/2}(vw)\|_{L^2} &\leq C\left(\|v\|_{L^\infty}\|w\|_{H^s}+\|w\|_{L^\infty}\|v\|_{H^s}\right) \\ &\leq C\|u\|_{H^s}\|v\|_{H^s}. \end{aligned}$$

Here, the constant C contains binomial coefficients and therefore is independant of L. Moreover, again with the embedding $H^s\hookrightarrow L^\infty$, we have

$$\|vw\|_{L^2}\leq\|v\|_{L^\infty}\|w\|_{L^2}\leq C\|u\|_{H^s}\|v\|_{H^s},$$

and the two last inequalities end the proof of the lemma. □

Therefore, coming back to (11), using the boundedness of E in H^s and Lemma 1, we have

$$\|u(t)\|_{H^s}\leq\|u_0\|_{H^s}+CT\|u(t)\|_{H^s}^3,$$

with $C>0$ independant of the period. This last estimate allows us to perform a Banach fixed point argument (the Lipschitz property is proved with similar arguments) in a ball of the space $\mathcal{C}([0,T],H^s)$ of radius $M=2\|u_0\|_{H^s}$ and with $T=C/\|u_0\|_{H^s}^2$, and this proves (9). □

Low regularity. This part is the more interesting since, as said above, the explicit blow up solution $v(t)$ of the introduction lives only in H^s with $s<1$. So let $s>1/2$. We define the Bourgain spaces associated with equation (8) as the completion of the space of smooth compactly supported functions on $\mathbb{R}\times T_L^2$ for the norm defined by:

$$\|u\|_{X_L^{s,b}}=\left\|\langle i\partial_t+P\rangle^b\langle(-\Delta)^{\frac{1}{2}}\rangle^s u\right\|_{L^2(\mathbb{R}\times T_L^2)},$$

where $\langle \alpha \rangle = (1+\alpha^2)^{1/2}$. Note that there exist more convenient equivalent definitions of this space: we may also check that the norm is equivalent to the following

$$\|u\|^2_{X_L^{s,b}} = \sum_{(m,n)\in\mathbb{Z}^2} \left(1+\frac{m^2}{L^2}+\frac{n^2}{L^2}\right)^s \int_{\mathbb{R}} \langle \tau - \frac{m^2}{L^2}+\frac{n^2}{L^2}\rangle^{2b} |\widehat{c_{m,n}}(\tau)|^2 d\tau,$$

where $\widehat{c_{m,n}}$ is the Fourier transform of $c_{m,n}$. A last definition is possible linking the Bourgain norm with Sobolev norm of the free dynamic:

$$\|u\|_{X_L^{s,b}} = \|e^{-itP}u(t)\|_{H^b(\mathbb{R},H^s(T_L^2))}. \tag{13}$$

We will work on a finite time interval so that we have to define the localized version of Bourgain spaces; for $u : [0,T]\times T_L^2 \to \mathbb{C}$:

$$\|u\|_{X_{L,T}^{s,b}} = \inf\left\{ \|v\|_{X_L^{s,b}},\ v \in X_L^{s,b} \text{ such that } v(t)=u(t) \text{ for all } t\in[0,T]\right\}. \tag{14}$$

Let us recall the integral formulation (10):

$$u(t) = e^{itP}u_0 + i\int_0^t e^{i(t-\tau)P}\left(|u(\tau)|^2u(\tau) + E(|u(\tau)|^2)u(\tau)\right)d\tau.$$

First, the source term is easily bounded: if $T\leq 1$ and $\psi(t)$ denotes a smooth real cut-off function equal to 1 on $[0,1]$ and with compact support, we get using the definition of Bourgain spaces (13):

$$\|e^{itP}u_0\|_{X_{L,T}^{s,b}} \leq \|e^{itP}\psi(t)u_0\|_{X_L^{s,b}} \leq \|\psi(t)u_0\|_{H^b(\mathbb{R},H^s(T_L^2))} \leq C\|u_0\|_{H^s(T_L^2)}, \tag{15}$$

where $C = \|\psi\|_{H^b(\mathbb{R})}$ is independant of the period L.

Lemma 2. *There exists $C>0$ such that for all $L\geq 1$, $T\leq 1$ and all couple (b,b') satisfying* $0<b'<1/2<b,\ b+b'<1$,

$$\left\|\int_0^t e^{i(t-\tau)P}F(\tau)d\tau\right\|_{X_{L,T}^{s,b}} \leq CT^{1-b-b'}\|F\|_{X_{L,T}^{s,-b'}}.$$

Proof. For a fixed $L>0$, this estimate is classical in the context of Bourgain spaces. To see that we may choose C independant of L, we remark (see [2]) that the proof of such an estimate for a fixed L relies on the one dimensional inequality (proved in [7]):

$$\|\phi(\frac{t}{T})\int_0^t g(\tau)d\tau\|_{H^b(\mathbb{R})} \leq CT^{1-b-b'}\|g\|_{H^{-b'}(\mathbb{R})}, \tag{16}$$

for a cut-off ϕ. Then we apply this estimate pointwise with $g(\tau) = (F(\tau,x),e_{m,n})e_{m,n}$, take the square, integrate on T_L^2, multiply by $(-m^2+n^2)/L^2$ and sum for $(m,n)\in\mathbb{Z}^2$. We then obtain the desired estimate with the same constant C as in (16) thus independant of L. □

Proposition 1 (**Uniform periodic bilinear Strichartz estimate**). *There exists $C>0$ such that for every $N_1, N_2 \geq 1$ dyadic numbers, $(a_1,b_1),(a_2,b_2) \in \mathbb{Z}^2$, $L \geq 1$ and $u_1, u_2 \in L^2(T_L^2)$ writing*

$$u_1 = \sum_{N_1 \leq \mathrm{Max}\left(\left|\frac{m}{L}-a_1\right|,\left|\frac{n}{L}-b_1\right|\right) \leq 2N_1} c_1(m,n) e_{m,n}, \qquad u_2 = \sum_{N_2 \leq \mathrm{Max}\left(\left|\frac{m}{L}-a_2\right|,\left|\frac{n}{L}-b_2\right|\right) \leq 2N_2} c_2(m,n) e_{m,n}$$

we have the bilinear estimate

$$\|e^{itP}(u_1)e^{itP}(u_2)\|_{L^2([0,1])L^2(T_L^2)} \leq C \min(N_1,N_2)^{1/2} \|u_1\|_{L^2(T_L^2)} \|u_2\|_{L^2(T_L^2)}. \tag{17}$$

Proof. Note that for $L=1$, linear Strichartz estimates has been proved recently in [16], [8]. We first prove the property in the case where $u_1 = u_2$ and $a_1 = b_1 = a_2 = b_2 = 0$. So let $u = u_1 = u_2$ and $N = N_1 = N_2$. We recall the semiclassical Strichartz estimate on the torus of size 1 (see [8]): for all $h \in (0,1)$, for all interval J of size h and for all u_0 writing

$$v_0 = \sum_{h^{-1} \leq \mathrm{Max}(|m|,|n|) \leq 2h^{-1}} c(m,n) e_{m,n},$$

for some coefficient $c(m,n)$, we have

$$\|e^{itP} v_0\|_{L^4(J)L^4(T_1^2)} \leq C \|v_0\|_{L^2(T_1^2)}. \tag{18}$$

Similarly to the case where P is the Laplace operator (see [9]), we apply a scaling argument on this estimate to derive a linear Strichartz estimate on T_L^2 on the time interval $[0,1]$. Let $u_0 \in L^2(T_L^2)$ localized in frequency in $[0,N]$ i.e.

$$u_0 = \sum_{N \leq \mathrm{Max}\left(\left|\frac{m}{L}\right|,\left|\frac{n}{L}\right|\right) \leq 2N} c(m,n) e_{m,n}, \tag{19}$$

and $v_0 \in L^2(T_1^2)$ defined by $v_0(x) = u_0(Lx)$. Then computing the $L^4([0,1])L^4(T_L^2)$ of $\exp(itP)u_0$ in term of v_0 and applying a change of variable, we get

$$\|e^{itP} u_0\|_{L^4([0,1])L^4(T_L^2)} = L \|e^{itP} v_0\|_{L^4([0,L^{-2}])L^4(T_1^2)}.$$

Remark that v_0 writes

$$v_0 = \sum_{LN \leq \mathrm{Max}(|m|,|n|) \leq 2LN} c(m,n) e_{m,n},$$

so that we may apply (18) with $h \sim LN$. We need to consider two cases. If $L \geq N$, then $[0,L^{-2}] \subset [0,(LN)^{-1}]$ and so

$$\|e^{itP} u_0\|_{L^4([0,1])L^4} \leq L \|e^{itP} v_0\|_{L^4([0,(LN)^{-1}])L^4} \leq CL \|v_0\|_{L^2} \leq C \|u_0\|_{L^2}.$$

If $L < N$, we write $[0,L^{-2}]$ as a reunion of intervals $[t_k, t_{k+1}]$ with $t_{k+1} - t_k \sim (LN)^{-1}$ and $k \sim N/L$. We apply (18) on each $[t_k, t_{k+1}]$ and this gives:

$$\|e^{itP} u_0\|_{L^4([0,1])L^4} \leq C \left(\frac{N}{L}\right)^{1/4} L \|v_0\|_{L^2} \leq C \left(\frac{N}{L}\right)^{1/4} \|u_0\|_{L^2}.$$

If $L \geq 1$, we may in particular summarize the two last inequalities as

$$\|e^{itP} u_0\|_{L^4([0,1])L^4} \leq C N^{1/4} \|u_0\|_{L^2},$$

and this proves (17) if $u_1 = u_2$ and $a_1 = b_1 = a_2 = b_2 = 0$. Now we treat the case $u = u_1 = u_2$ but without the assumption $a_1 = b_1 = a_2 = b_2 = 0$. Let $(a, b) \in \mathbb{Z}^2$ and write

$$\begin{aligned} u &= \sum_{N \leq \mathrm{Max}\left(\left|\frac{m}{L}-a\right|,\left|\frac{n}{L}-b\right|\right) \leq 2N} e^{\frac{it}{L^2}(n^2-m^2)} c(m,n) e_{m,n} \\ &= e^{iax} e^{iby} e^{it(a^2-b^2)} \sum_{N \leq \mathrm{Max}\left(\left|\frac{p}{L}\right|,\left|\frac{q}{L}\right|\right) \leq 2N} c(aL+p, bL+q) e^{\frac{-it}{L^2}(p^2-q^2+2aLp-2bLq)} e_{p,q}. \end{aligned}$$

Then

$$\begin{aligned} \|u\|^4_{L^4L^4} &= \Big\| \sum_{N \leq \mathrm{Max}\left(\left|\frac{p}{L}\right|,\left|\frac{q}{L}\right|\right) \leq 2N} \frac{1}{2\pi L} c(aL+p, bL+q) e^{\frac{-it}{L^2}(p^2-q^2)} e^{\frac{ip}{L}(x-2at)} e^{\frac{iq}{L}(y+2bt)} \Big\|^4_{L^4L^4} \\ &= \int_t \int_{\substack{-2at \leq \alpha \leq 2\pi L - 2at \\ 2bt \leq \beta \leq 2\pi L + 2bt}} \left| \sum_{N \leq \mathrm{Max}\left(\left|\frac{p}{L}\right|,\left|\frac{q}{L}\right|\right) \leq 2N} e^{\frac{-it}{L^2}(p^2-q^2)} \frac{1}{2\pi L} c(aL+p, bL+q) e^{\frac{i}{L}p\alpha} e^{\frac{i}{L}q\beta} \right|^4 d\alpha d\beta dt \\ &= \int_t \int_{\substack{0 \leq \alpha \leq 2\pi L \\ 0 \leq \beta \leq 2\pi L}} \left| \sum_{N \leq \mathrm{Max}\left(\left|\frac{p}{L}\right|,\left|\frac{q}{L}\right|\right) \leq 2N} e^{\frac{it}{L^2}(q^2-p^2)} c(aL+p, bL+q) e_{p,q}(\alpha, \beta) \right|^4 d\alpha d\beta dt. \end{aligned}$$

We apply the linear result proved above with $(a, b) = (0, 0)$ and this gives

$$\begin{aligned} \|u\|^4_{L^4L^4} &\leq CN \left(\sum_{u,v} |c(u,v)|^2 \right)^2 \\ &\leq CN \|u_0\|^4_{L^2}. \end{aligned}$$

This proves the result when $u_1 = u_2$. Note that if we assume another type of localization for u

$$u = \sum_{\mathrm{Max}(|\frac{m}{L}-a|,|\frac{n}{L}-b|) \leq N} c(m,n) e_{m,n},$$

the L^4L^4 estimate still holds. It may be seen by remarking that estimate (18) also holds if u_0 is spectrally localized in $\{(m,n) \in \mathbb{Z}^2, \mathrm{Max}(|m|,|n|) \leq 2h^{-1}\}$ (see [8]) and using the same analysis as above. Now, we can prove the bilinear estimate in the general case. We assume for instance $N_1 \leq N_2$ and decompose the set $A = \{(m,n) \in \mathbb{Z}^2, N_2 \leq \mathrm{Max}(|a_2 - m/L|, |b_2 - n/L|) \leq 2N_2\}$ in small disjoint cubes of the form $Q_\alpha = Q_{(u,v)} = \{(m,n) \in A, \mathrm{Max}(|u - m/L|, |v - n/L|) \leq N_1\}$ for $\alpha = (u,v)$ running over a set I. Then for different $\alpha's$, the functions $e^{itP}(u_0)e^{itP}(\Delta_{Q_\alpha} v_0)$ are almost orthogonal since each function is localized in Fourier in the set $D_\alpha := \{(m,n) \in \mathbb{Z}^2, N_1 \leq \mathrm{Max}(|m/L - a_1|, |n/L - b_1|) \leq 2N_1\} + Q_\alpha$ and the sets D_α are almost disjoint in the sense that each point of $\mathbb{Z}^2$ belongs to a finite number of sets D_α. Indeed, if $(m,n) \in D_{\alpha_1} \cap D_{\alpha_2}$, then in particular we may write

$$(m,n) = c + d = e + f,$$

with $d \in Q_{\alpha_1}$, $f \in Q_{\alpha_2}$, and $c, e \in \{(m,n) \in \mathbb{Z}^2, N_1 \leq \text{Max}(|m/L - a_1|, |n/L - b_1|) \leq 2N_1\}$. We deduce $|c_1 - e_1| = |f_1 - d_1| \leq 4N_1 L$. But each Q_α is of size less than $4N_1 L$ and there is a finite number of Q_α whose distance to Q_{α_2} is less than $4N_1 L$. So if we fix α_1, then α_2 runs in a finite number of indexes. Thus, this orthogonality property implies

$$\begin{aligned}
\|e^{itP}(u_1)e^{itP}(u_2)\|^2_{L^2L^2} &\leq C\sum_{\alpha\in I}\|e^{itP}(u_1)e^{itP}(\Delta_{Q_\alpha}u_2)\|^2_{L^2L^2} \\
&\leq C\|e^{itP}(u_1)\|^2_{L^4L^4}\sum_{\alpha\in I}\|e^{itP}(\Delta_{Q_\alpha}u_2)\|^2_{L^4L^4} \\
&\leq CN_1^{1/2}\|u_1\|^2_{L^2}N_1^{1/2}\sum_{\alpha\in I}\|\Delta_{Q_\alpha}u_2\|^2_{L^2} \\
&\leq CN_1\|u_1\|^2_{L^2}\|u_2\|^2_{L^2}.
\end{aligned}$$

This proves the proposition. □

Remark 2. Note that if Q_i denotes the set

$$Q_i = \left\{(m,n) \in \mathbb{Z}^2, N_i \leq \text{Max}\left(\left|\frac{m}{L} - a_i\right|, \left|\frac{n}{L} - b_i\right|\right) \leq 2N_i\right\},$$

then $(N_i L)^2 \leq |Q_i|$ and we may rewrite the Strichartz estimate (17) as

$$\|\Delta_{Q_1}(e^{itP}u)\Delta_{Q_2}(e^{itP}v)\|_{L^2L^2} \leq C\left(\frac{\min(|Q_1|,|Q_2|)}{L^2}\right)^{\frac{1}{4}}\|u\|_{L^2}\|v\|_{L^2}. \tag{20}$$

Once we have proved (20), from recovering arguments, we may deduce the same estimate for other shapes of Q_i typically $Q_i = \{(m,n) \in \mathbb{Z}^2, \text{Max}(|m/L|, |n/L|) \leq 2N_i\}$ or translated sets of the previous one. In the sequel, we will use (20) (or more precisely the following lemma) for these kinds of Q_i.

Lemma 3. *For all $b > 1/2$, there exists $C(b) > 0$, $\beta(b) \in (0, 1-b)$ and $\varepsilon(b) > 0$ such that for all dyadic square $Q_1, Q_2 \subset \mathbb{Z}^2$, R_1, R_2 dyadic number, $L \geq 1$ and $u_0, v_0 \in L^2(\mathbb{R}, L^2(T_L^2))$,*

$$\begin{aligned}
\|\Delta_{Q_1,R_1}u_0\ \Delta_{Q_2,R_2}v_0\|_{L^2L^2} \leq C(b)&\left(\frac{\text{Min}(|Q_1|,|Q_2|)}{L^2}\right)^{1/4+\varepsilon(b)}(R_1R_2)^{\beta(b)} \\
&\times\|\Delta_{Q_1,R_1}u_0\|_{L^2L^2}\|\Delta_{Q_2,R_2}v_0\|_{L^2L^2},
\end{aligned}\tag{21}$$

where $|Q_i|$ denotes the number of points in Q_i. Moreover, we may choose $\varepsilon(b)$ such that $\varepsilon(b)$ goes to 0 as b goes to $1/2$.

Proof. As for the proof of (17), we first assume $u = u_0 = v_0$. Next it is a classical result that bilinear estimates on the flow such as (17) may be translated in terms of Bourgain spaces. Thus, following [2], we get for all $b > 1/2$ and $f \in X_L^{0,b}$ localized in frequency in Q,

$$\|f\|_{L^4L^4} \leq C\left(\frac{|Q|}{L^2}\right)^{1/8}\|f\|_{X_L^{0,b}}.$$

Again the constant C does not depend on L since the proof relies on manipulations only in time. In particular, for all u,

$$\|\Delta_{Q,R}u\|_{L^4L^4} \leq C\left(\frac{|Q|}{L^2}\right)^{1/8}\|\Delta_{Q,R}u\|_{X_L^{0,b}}, \tag{22}$$

for all $b > 1/2$. And this gives using properties of Bourgain spaces

$$\|\Delta_{Q,R}u\|_{L^4L^4} \leq C\left(\frac{|Q|}{L^2}\right)^{1/8} R^b\|\Delta_{Q,R}u\|_{L^2L^2}. \tag{23}$$

The fact that $b > 1/2$ in the above estimate will not be enough to conclude so that we need to refine this L^4L^4 estimate. To do so, we compute the $L^\infty L^\infty$ norm of $\Delta_{Q,R}u$. From the definition of the projector $\Delta_{Q,R}$, we get using twice Cauchy-Schwartz

$$\begin{aligned}
\|\Delta_{Q,R}u\|_{L^\infty L^\infty} &\leq \frac{1}{L}\sum_{(m,n)\in Q}\int_{R\leq\langle\tau-\frac{m^2}{L^2}+\frac{n^2}{L^2}\rangle\leq 2R}|\widehat{c_{m,n}}(\tau)|d\tau \\
&\leq \frac{R^{1/2}}{L}\sum_{(m,n)\in Q}\left(\int_{R\leq\langle\tau-\frac{m^2}{L^2}+\frac{n^2}{L^2}\rangle\leq 2R}|\widehat{c_{m,n}}(\tau)|^2d\tau\right)^{1/2} \\
&\leq R^{1/2}\left(\frac{|Q|}{L^2}\right)^{1/2}\left(\sum_{(m,n)\in Q}\int_{R\leq\langle\tau-\frac{m^2}{L^2}+\frac{n^2}{L^2}\rangle\leq 2R}|\widehat{c_{m,n}}(\tau)|^2d\tau\right)^{1/2} \\
&\leq \left(\frac{|Q|}{L^2}\right)^{1/2}R^{1/2}\|\Delta_{Q,R}u\|_{L^2L^2}. \qquad (24)
\end{aligned}$$

By interpolation between the trivial inequality $\|\Delta_{Q,R}u\|_{L^2L^2} \leq \|\Delta_{Q,R}u\|_{L^2L^2}$ and (24), we have

$$\|\Delta_{Q,R}u\|_{L^4L^4} \leq \left(\frac{|Q|}{L^2}\right)^{1/4} R^{1/4}\|\Delta_{Q,R}u\|_{L^2L^2}. \tag{25}$$

Let $\varepsilon(b) > 0$ such that $\delta(b) := b(1-8\varepsilon(b)) + 8\varepsilon(b)\frac{1}{4} \in (0, 1-b)$ and $\varepsilon(b) \to 0$ as $b \to 1/2$. For instance choose $\delta(b) = 3/2 - 2b$. Next, by interpolation between (23) with weight $1-8\varepsilon(b)$ and (25) with weight $8\varepsilon(b)$, we get the expected estimate:

$$\|\Delta_{Q,R}u_0\|_{L^4L^4} \leq C\left(\frac{|Q|}{L^2}\right)^{1/8+\varepsilon(b)} R^{\delta(b)}\|\Delta_{Q,R}u_0\|_{L^2L^2}. \tag{26}$$

To deduce (21) from (26), we proceed as in the proof of Strichartz estimate (17) and decompose the biggest square in pieces of size the smallest and next apply almost orthogonality argument. We omit this argument and the proof is over. □

Proposition 2 (**Trilinear estimate**). *There exists a constant $C > 0$ such that for every $L \geq 1, T > 0$, $u_1, u_2, u_3 \in X_{L,T}^{s,b}$,*

$$\|u_1u_2u_3\|_{X_{L,T}^{s,-b'}} \leq C\|u_1\|_{X_{L,T}^{s,b}}\|u_2\|_{X_{L,T}^{s,b}}\|u_3\|_{X_{L,T}^{s,b}},$$

$$\|E(u_1u_2)u_3\|_{X^{s,-b'}_{L,T}} \leq C\|u_1\|_{X^{s,b}_{L,T}}\|u_2\|_{X^{s,b}_{L,T}}\|u_3\|_{X^{s,b}_{L,T}}.$$

Proof. It is enough to prove the trilinear estimate for the global space $X_L^{s,b}$ i.e. $T=\infty$, then we recover the local in time estimate by taking the infimum on all extensions of $u_1,u_2,u_3 \in X^{s,b}_{L,T}$. Moreover, we only prove the second estimate; the first one being easier. By a duality argument, we have to show the quadrilinear estimate: there exists $C>0$ such that for all $L\geq 1$, $u_1,u_2,u_3,u_4 \in X_L^{s,b}$:

$$\left|\int_{\mathbb{R}\times T_L^2} E(u_1u_2)u_3u_4\right| \leq C\|u_1\|_{X_L^{s,b}}\|u_2\|_{X_L^{s,b}}\|u_3\|_{X_L^{s,b}}\|u_4\|_{X_L^{-s,b'}}.$$

In the sequel, we will note $Q_i=\{(m,n)\in\mathbb{Z}^2, N_i\leq \text{Max}(|m/L|,|n/L|)<2N_i\}$. Decomposing each u_i as

$$u_i=\sum_{N_i,R_i}\Delta_{N_i,R_i}(u_i),$$

we have that

$$G=\int_{\mathbb{R}\times T_L^2} E(u_1u_2)u_3u_4$$

becomes

$$G=\int_{\mathbb{R}\times T_L^2}\sum_{\substack{N_1,N_2,N_3,N_4\\R_1,R_2,R_3,R_4}} E\left(\Delta_{N_1,R_1}(u_1)\Delta_{N_2,R_2}(u_2)\right)\Delta_{N_3,R_3}(u_3)\Delta_{N_4,R_4}(u_4).$$

In the summation above, we may restrict indexes to $N_4\leq 2(N_1+N_2+N_3)$. Indeed, the function

$$U=E\left(\Delta_{N_1,R_1}(u_1)\Delta_{N_2,R_2}(u_2)\right)\Delta_{N_3,R_3}(u_3)$$

is localized in Fourier in the set $\{(m,n)\in\mathbb{Z}^2, m=m_1+m_2+m_3, n=n_1+n_2+n_3, (m_1,n_1)\in Q_1, (m_2,n_2)\in Q_2, (m_3,n_3)\in Q_3\}$. Thus, if $N_4>2(N_1+N_2+N_3)$, the integral over T_L^2 of $U\Delta_{N_4,R_4}(u_4)$ is zero. Therefore

$$G=\sum_{\substack{N_4\leq 2(N_1+N_2+N_3)\\R_1,R_2,R_3,R_4}}\alpha(N_1,N_2,N_3,N_4,R_1,R_2,R_3,R_4), \tag{27}$$

where

$$\alpha(N_1,N_2,N_3,N_4,R_1,R_2,R_3,R_4)=\int_{\mathbb{R}\times T_L^2} E\left(\Delta_{N_1,R_1}(u_1)\Delta_{N_2,R_2}(u_2)\right)\Delta_{N_3,R_3}(u_3)\Delta_{N_4,R_4}(u_4),$$

Contrary to the case of a typical cubic nonlinearity, α is not symmetric in N_1,N_2,N_3,N_4 and we need to split the analysis. The worst situation is when the two lowest frequencies appear in the nonlocal term. Let us first treat this case.

Case $N_3=\max(N_1,N_2,N_3)$. Without loss of generality, we may assume $N_1\leq N_2\leq N_3$. In this situation, we decompose the set Q_3 in small pieces of size N_2L. Hence, we may write Q_3 as a disjoint union of sets of the form $Q_\alpha=Q_{(a,b)}=\{(m,n)\in Q_3, \text{Max}(|a-$

$m/L|, |b-n/L|) \leq N_2\}$ for some well chosen set I of couples $\alpha = (a,b) \in Q_3$ so that the union is disjoint. Using again an orthogonality argument, α is then

$$\alpha(N_i, R_i) = \int_{\mathbb{R}\times T_L^2} E\left(\Delta_{N_1,R_1}(u_1)\Delta_{N_2,R_2}(u_2)\right)\Delta_{Q_\alpha,R_3}(u_3)\Delta_{\tilde{Q}_\alpha,R_4}(u_4)$$

where

$\tilde{Q}_\alpha = \{(m_4,n_4) \in Q_4, m = -m_1-m_2-m_3, n = -n_1-n_2-n_3, (m_i,n_i) \in Q_i, i = 1,2, (m_3,n_3) \in Q_\alpha\}$.

From Cauchy-Schwarz in space and time and the boundedness of E on $L^2(T_L^2)$,

$$|\alpha(N_i,R_i)| \leq \|\Delta_{N_1,R_1}(u_1)\Delta_{N_2,R_2}(u_2)\|_{L^2L^2}\|\Delta_{Q_\alpha,R_3}(u_3)\Delta_{\tilde{Q}_\alpha,R_4}(u_4)\|_{L^2L^2}.$$

Note that since $|Q_\alpha| \leq (N_2L)^2$, we deduce by triangle inequality that we also have $|\tilde{Q}_\alpha| \leq C(LN_2)^2$ and thus we can apply Lemma 3 to get

$$\begin{aligned}\alpha(N_i,R_i) \leq CN_1^{\frac{1}{2}+\varepsilon(b)}N_2^{\frac{1}{2}+\varepsilon(b)}&(R_1R_2R_3R_4)^{\beta(b)}\|\Delta_{N_1,R_1}(u_1)\|_{L^2L^2}\\ &\times\|\Delta_{N_2,R_2}(u_1)\|_{L^2L^2}\sum_{\alpha\in I}\|\Delta_{Q_\alpha,R_3}(u_3)\|_{L^2L^2}\|\Delta_{\tilde{Q}_\alpha,R_4}(u_4)\|_{L^2L^2}.\end{aligned} \tag{28}$$

Next from Cauchy-Schwarz, we may write

$$\sum_{\alpha\in I}\|\Delta_{Q_\alpha,R_3}(u_3)\|_{L^2L^2}\|\Delta_{\tilde{Q}_\alpha,R_0}(u_0)\|_{L^2L^2} \leq \left(\sum_{\alpha\in I}\|\Delta_{Q_\alpha,R_3}(u_3)\|^2_{L^2L^2}\right)^{\frac{1}{2}}\left(\sum_{\alpha\in I}\|\Delta_{\tilde{Q}_\alpha,R_4}(u_4)\|^2_{L^2L^2}\right)^{\frac{1}{2}}.$$

First, since $(Q_\alpha)_\alpha$ is a partition of Q_3, by orthogonality, we have for the first term in the right hand side above:

$$\left(\sum_{\alpha\in I}\|\Delta_{Q_\alpha,R_3}(u_3)\|^2_{L^2L^2}\right)^{1/2} = \|\Delta_{Q_3,R_3}(u_3)\|_{L^2L^2}.$$

For the second term, the $\tilde{Q}_\alpha$'s recover Q_4 but since there are not disjoint, strict orthogonality is broken. However, using the same argument of almost orthogonality as for the proof of Strichartz estimate (each point of Q_4 belongs to a finite number of $\tilde{Q}_\alpha$), we deduce

$$\left(\sum_{\alpha\in I}\|\Delta_{\tilde{Q}_\alpha,R_4}(u_4)\|^2_{L^2L^2}\right)^{1/2} \leq C\|\Delta_{Q_4,R_4}(u_0)\|_{L^2L^2}.$$

Thus,

$$\alpha(N_i,R_i) \leq C(N_1N_2)^{1/2+\varepsilon(b)}(R_1R_2R_3R_4)^{\beta(b)}\prod_{i=0}^{3}\|\Delta_{Q_i,R_i}(u_i)\|_{L^2L^2}. \tag{29}$$

We reorder terms to make appear Bourgain's norms of u_i. The quantity

$$H = \sum_{\substack{N_4\leq 2(N_1+N_2+N_3)\\ R_1,R_2,R_3,R_4\\ N_3=\mathrm{Max}(N_1,N_2,N_3)}}\alpha(N_i,R_i)$$

is bounded by

$$\begin{aligned}
|H| \quad \leq \quad & \sum_{N_1,R_1} N_1^{\frac{1}{2}+\varepsilon(b)-s} R_1^{\beta(b)-b} N_1^s R_1^b \|\Delta_{N_1,R_1}(u_1)\|_{L^2L^2} \\
& \times \sum_{N_2,R_2} N_2^{\frac{1}{2}+\varepsilon(b)-s} R_2^{\beta(b)-b} N_1^s R_1^b \|\Delta_{N_2,R_2}(u_2)\|_{L^2L^2} \\
& \times \sum_{N_4 \leq 6N_3} \sum_{R_4,R_3} R_0^{\beta(b)-b'} R_4^{b'} R_3^{\beta(b)-b} R_3^b \|\Delta_{N_4,R_4}(u_4)\|_{L^2L^2} \|\Delta_{N_3,R_3}(u_3)\|_{L^2L^2}.
\end{aligned}$$

For the first two sums above, we use Cauchy-Schwarz to recover Bourgain norm of u_i. For instance for the first term, we have if $s > 1/2 + \varepsilon(b)$, and since $b > \beta(b)$,

$$\begin{aligned}
\sum_{N_1,R_1} N_1^{\frac{1}{2}+\varepsilon(b)-s} R_1^{\beta(b)-b} N_1^s R_1^b \|\Delta_{N_1,R_1}(u_1)\|_{L^2L^2} \quad & \leq \quad \|u_1\|_{X_L^{s,b}} \left(\sum_{N_1,R_1} N_1^{1+2\varepsilon(b)-2s} R_1^{2(\beta(b)-b)} \right)^{\frac{1}{2}} \\
& \leq \quad C\|u_1\|_{X_L^{s,b}}.
\end{aligned}$$

For the third sum, using again Cauchy-Schwartz inequality, and choosing $b' > \beta(b)$ (this condition is compatible with $1 - b - b' > 0$ since $\beta(b) < 1 - b$), we write

$$\begin{aligned}
\sum_{R_4} R_4^{\beta(b)-b'} R_4^{b'} \|\Delta_{N_4,R_4}(u_4)\|_{L^2L^2} \quad & \leq \quad \left(\sum_{R_4} R_4^{2\beta(b)-2b'} \right)^{\frac{1}{2}} \left(\sum_{R_4} R_4^{2b'} \|\Delta_{N_4,R_4}(u_4)\|_{L^2L^2}^2 \right)^{\frac{1}{2}} \\
& \leq \quad C\|\Delta_{N_4}(u_4)\|_{X_L^{0,b'}}.
\end{aligned}$$

We treat the sum over R_3 in the same way. Therefore,

$$|H| \leq \|u_1\|_{X_L^{s,b}} \|u_2\|_{X_L^{s,b}} \sum_{N_4 \leq 6N_3} \frac{N_4^s}{N_3^s} N_4^{-s} \|\Delta_{N_4}(u_4)\|_{X_L^{0,b'}} N_3^s \|\Delta_{N_3}(u_3)\|_{X_L^{0,b}}.$$

Now we need the following lemma (see [3] for a proof) to conclude.

Lemma 4. *For every $s > 0$, there exists a constant $C > 0$ such that for all sequence $(a_{N_4})_{N_4 \in 2^{\mathbb{N}}}, (b_{N_3})_{N_3 \in 2^{\mathbb{N}}}$, we have*

$$\sum_{N_4 \leq 6N_3} \left(\frac{N_4}{N_3} \right)^s |a_{N_4} b_{N_3}| \leq C \left(\sum_{N_4} a_{N_4}^2 \right)^{1/2} \left(\sum_{N_3} a_{N_3}^2 \right)^{1/2}.$$

To conclude in this case, we apply the lemma with

$$a_{N_4} = N_4^{-s} \|\Delta_{N_4}(u_4)\|_{X_L^{0,b'}}, \quad b_{N_3} = N_3^s \|\Delta_{N_3}(u_3)\|_{X_L^{0,b}},$$

and obtain

$$|H| \leq C\|u_1\|_{X_L^{s,b}} \|u_2\|_{X_L^{s,b}} \|u_3\|_{X_L^{s,b}} \|u_4\|_{X_L^{-s,b'}}. \tag{30}$$

Case $N_3 < \max(N_1, N_2, N_3)$. In the summation (27), we assume for instance $N_1 \le N_3 \le N_2$. This case is easier since we do not need to decompose high frequencies. With the definition of $\alpha(N_i, R_i)$ and from Cauchy-Schwarz:

$$|\alpha(N_i, R_i)| \le \|\Delta_{N_1,R_1}(u_1)\Delta_{N_2,R_2}(u_2)\|_{L^2L^2}\|\Delta_{N_3,R_3}(u_3)\Delta_{N_4,R_4}(u_4)\|_{L^2L^2}.$$

Coming back to Lemma 3, we have directly the equivalent of (29) that is

$$\alpha(N_i, R_i) \le (N_1N_3)^{1/2+\varepsilon}(R_4R_1R_2R_3)^{\beta(\varepsilon)}\prod_{i=1}^{4}\|\Delta_{N_i,R_i}(u_i)\|_{L^2L^2}.$$

Once we have this estimate, the end of the proof in this case is the same as the previous one and we obtain

$$\sum_{\substack{N_3<\max(N_1,N_2,N_3)\\ R_1,R_2,R_3,R_4\\ N_4\le 6\max(N_1,N_2,N_3)}} \alpha(N_i, R_i) \le C\|u_1\|_{X_L^{s,b}}\|u_2\|_{X_L^{s,b}}\|u_3\|_{X_L^{s,b}}\|u_4\|_{X_L^{-s,b'}}. \tag{31}$$

Estimates (30) and (31) provides Proposition 2. □

Writing the Duhamel formula (10) and using (15), Lemma 16 and Proposition 2, we have easily

$$\|\Phi(u)\|_{X_{L,T}^{s,b}} \le C\|u_0\|_{H^s} + CT^{1-b-b'}\|u\|^3_{X_{L,T}^{s,b}},$$

and

$$\|\Phi(u) - \Phi(v)\|_{X_{L,T}^{s,b}} \le CT^{1-b-b'}\left(\|u\|^2_{X_{L,T}^{s,b}} + \|v\|^2_{X_{L,T}^{s,b}}\right)\|u-v\|_{X_{L,T}^{s,b}},$$

with

$$\Phi(u) = e^{itP}u_0 + i\int_0^t e^{i(t-\tau)P}\left(|u(\tau)|^2u(\tau) + E(|u(\tau)|^2)u(\tau)\right)d\tau.$$

Therefore, we may close the fixed point argument in the ball $B(0,R)$ of $X_{L,T}^{s,b}$ with $M = 2C\|u_0\|_{H^s}$ and $T \ge D/\|u_0\|_{H^s}^{2/(1-b-b')}$ with $D > 0$ independant of the period $L \ge 1$. This proves (9) for low regularities and also the first point (take $L = 1$) in Theorem 1.

Step 2. Let us now finish the proof of the lower bound (6). Let $u \in H^s(T^2)$ solution to (2) and consider the family for $\tau \in [0, T)$:

$$v^\tau(t, x, y) = \lambda(\tau)u(\lambda^2(\tau)t + \tau, \lambda(\tau)x, \lambda(\tau)y),$$

where $\lambda(\tau) = \|u(\tau)\|_{H^s(T^2)}^{-1/s}$. For all τ, v^τ is a function on the torus $T_{1/\lambda(\tau)}$ and satifies the equation (8) for $L = 1/\lambda(\tau)$. Moreover, it is easy to check that $\|v^\tau(0)\|_{L^2} = \|u(0)\|_{L^2}$ and $\|(-\Delta)^{s/2}(v^\tau(0))\|_{L^2} \le 1$. If we denote by T_τ the maximal time for v^τ, from (9), we deduce the uniform bound, $T_\tau \ge C > 0$. But $T_\tau = (T-\tau)/\lambda^2(\tau)$ where T is the maximal time for u and this with the uniform lower bound on T_τ proves the lower bound (6).

Acknowledgements. I would like to thank Nikolay Tzvetkov for his guidance and helpful advices.

References

[1] V. Barros. The Davey Stewartson system in weak L^p spaces. *Arxiv preprint arXiv:1104.1454*, 2011.

[2] N. Burq, P. Gérard, and N. Tzvetkov. Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces. *Inventiones mathematicae*, 159(1):187–223, 2005.

[3] N. Burq, P. Gérard, and N. Tzvetkov. Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations. In *Annales Scientifiques de l'École Normale Supérieure*, volume 38, pages 255–301. Elsevier, 2005.

[4] H. Chihara. The initial value problem for the elliptic-hyperbolic Davey-Stewartson equation. *J. Math. Kyoto Univ.*, 39(1):41–66, 1999.

[5] J.-M. Ghidaglia and J.-C. Saut. On the initial value problem for the Davey-Stewartson systems. *Nonlinearity*, 3(2):475–506, 1990.

[6] J.-M. Ghidaglia and J.-C. Saut. Nonexistence of travelling wave solutions to nonelliptic nonlinear Schrödinger equations. *J. Nonlinear Sci.*, 6(2):139–145, 1996.

[7] J. Ginibre. Le probleme de cauchy pour des EDP semi-linéaires périodiques en variables d'espace. *Séminaire Bourbaki*, 796:163–187, 1995.

[8] N. Godet and N. Tzvetkov. Strichartz estimates for the periodic non elliptic Schrödinger equation. *Arxiv preprint arXiv:1207.0213*, 2012.

[9] Z. Hani. Global well-posedness of the cubic nonlinear Schrödinger equation on closed manifolds. *Communications in Partial Differential Equations*, 37(7):1186–1236, 2012.

[10] N. Hayashi. Local existence in time of solutions to the elliptic-hyperbolic Davey-Stewartson system without smallness condition on the data. *J. Anal. Math.*, 73:133–164, 1997.

[11] C. Klein, B. Muite, and K. Roidot. Numerical study of blowup in the Davey-Stewartson system. *Arxiv preprint arXiv:1112.4043*, 2011.

[12] Y. Li and M. Guo, B.and Jiang. Existence and blow-up of solutions to degenerate Davey-Stewartson equations. *J. Math. Phys.*, 41(5):2943–2956, 2000.

[13] F. Linares and G. Ponce. On the Davey-Stewartson systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(5):523–548, 1993.

[14] T. Ozawa. Exact blow-up solutions to the Cauchy problem for the Davey-Stewartson systems. *Proc. Roy. Soc. London Ser. A*, 436(1897):345–349, 1992.

[15] G. Richards. Mass concentration for the Davey-Stewartson system. *Differential Integral Equations*, 24(3-4):261–280, 2011.

[16] Y. Wang. Periodic cubic hyperbolic Schrödinger equation on T^2. *Arxiv preprint arXiv:1205.5205*, 2012.

[17] J. Zhang and S. Zhu. Sharp blow-up criteria for the Davey-Stewartson system in R^3. *Dynamics of PDE*, 8(3):239–260, 2011.

University of Cergy-Pontoise, Department of Mathematics, CNRS, UMR 8088, F-95000 Cergy-Pontoise

E-mail address: nicolas.godet@u-cergy.fr